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Describing rotations of the spheroids based on the first order polarization tensor

Nurhazirah Mohd Yunos and Taufiq Khairi Ahmad Khairuddin
Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 Johor Bahru, Johor, Malaysia.
UTM Centre for Industrial and Applied Mathematics, Universiti Teknologi Malaysia, 81310 Johor Bahru, Johor, Malaysia.
Corresponding Author: taufiq@utm.my

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ABSTRACT

Polarization Tensor (PT) can be used to describe conducting objects presented in electric or electromagnetic fields. In this paper, we investigate specifically the first order PT when the conducting object is a spheroid. First of all, we evaluate the first order PT for a prolate and an oblate spheroid. After that, by using these first order PT and a specific matrix transformation, we numerically present some transformation of the first order PT representing the first order PT after the spheroids are being rotated. Moreover, it is found that the first order PT for the spheroid before it is being rotated and after it is being rotated actually have the same determinant.

Keywords: Conductivity; Rotation Matrix; Determinant.

1. INTRODUCTION

Polarization tensor (PT) is an old terminology in mathematics and physics. It is used generally to describe the perturbation in electromagnetic or electric field in many real problems such as electrical imaging [1], material science [2], electrosensing fish [3,4,5] and also metal detection [4,6,7]. Ammari and Kang [1] have specifically shown that the perturbation in the electric field due to the presence of conducting objects can be represented by an asymptotic formula where, the dominant term of the formula can be expressed in terms of the PT called as Generalized Polarization Tensor (GPT). Here, the first order PT is actually included in GPT. Besides, we simply called the first order PT in the asymptotic formula representing the perturbation due to the presence of conducting object B as the first order PT for B. Generally, the first order PT for B can be computed by numerically solving integral equations, as described in [8,9]. However, if the object B is an ellipsoid, the simpler explicit formula as given by Ammari and Kang [1] can be used to compute the first order PT.

In order to calculate the first order PT for ellipsoid, the integrals given by Ammari and Kang [1] must be evaluated. By appropriate substitution as stated in [9], these integrals can be reformulated as the depolarization factors for ellipsoid with semi principal axes a, b and c. Moreover, Milton [2] has shown that the depolarization factors are also equivalent to the classical integrals in [10,11]. For spheroid ellipsoids, the depolarization factors can be further reduced to simpler formulas. Similarly, by using these formulas, the first order PT for spheroids can also be further simplified.

In this paper, we are interested in studying the first order PT when an electrical field is perturbed by a conducting spheroid. Based on the new explicit formula of the first order PT for spheroid which includes the depolarization factors for spheroid, we will investigate some transformations to the first order PT for a spheroid after the spheroid is rotated. We focus on spheroid as previous study [3,9,12] have shown that the first order PT for many objects actually can be related to the first order PT for a spheroid.

2. THE FIRST ORDER PT FOR SPHEROID

Let B be an ellipsoid with semi principal axes a, b and c represented by $\left(\frac{x}{a}\right)^2+\left(\frac{y^2+z^2}{b^2}\right)=1$ in the Cartesian coordinate system. The first order PT for B is given explicitly by [1] as

$$M(k,B)=(k-1)\,|B|\begin{pmatrix} \frac{1}{(1-d_1)+kd_1} & 0 & 0 \\ 0 & \frac{1}{(1-d_2)+kd_2} & 0 \\ 0 & 0 & \frac{1}{(1-d_3)+kd_3} \end{pmatrix} \quad (1)$$

where |*B*| is the volume of *B* and $0 < k \neq 1 < +\infty$. The depolarization factor, d_1 is given by

$$d_1(a,b) = \frac{ab^2}{2}\int_0^{\infty}\frac{1}{(a^2+y)^{3/2}(b^2+y)}\,dy. \tag{2}$$

Moreover, according to Milton [2] and Stoner [11], if *B* is a prolate spheroid $(a \geq b, b = c)$, then d_1 is simplified to

$$d_1 = \frac{1-\varepsilon^2}{\varepsilon^2}\left(\frac{1}{2\varepsilon}\ln\left(\frac{1+\varepsilon}{1-\varepsilon}\right)-1\right), \tag{3}$$

where $\varepsilon = \sqrt{1-\left(\frac{b}{a}\right)^2}$. Similarly, if *B* is an oblate spheroid $(a \leq b, b = c)$, d_1 can be reduced to

$$d_1 = \frac{1}{\psi^2}\left(1-\frac{\sqrt{1-\psi^2}}{\psi}\sin^{-1}\psi\right), \tag{4}$$

where $\psi = \sqrt{1-\left(\frac{a}{b}\right)^2}$.

3. SOME PROPERTIES OF THE FIRST ORDER PT FOR SPHEROID

If *B* is a spheroid, the first order PT for *B* at a fixed *k* can be obtained by using (1) and (2) (or either (3) or (4)) where, *k* represents the material of *B*. The resulting matrix actually depends on the value of *k*. This is explained in Ammari and Kang [1] in the following proposition.

Proposition 1 The first order PT is positive definite if $k > 1$ and it is negative definite if $k < 1$.

In this paper, a matrix is defined as a positive definite if all of its eigenvalues are positive whereas it is negative if all of its eigenvalues are negative. Moreover, according to Ammari and Kang [1], if the spheroid is rotated around coordinate axes, the first order PT for the spheroid after it is rotated can be determined from the first order PT for the spheroid before it is rotated. The method to find the first order PT in this case is described in the next proposition, given by [1].

Proposition 2 Let B' be a domain and $B = RB'$ where R is a unitary transformation and R^T is the transpose of R. Let $M(k,B)$ and $M(k,B')$ be the first order PT associated with B and B' at any k where $0 < k \neq 1 < +\infty$. Then, $R^T M(k,B) R = M(k,B')$.

In our case later on, by using rotation in three dimension as our unitary matrix transformation, we say that $M(k,B')$ is the result after $M(k,B)$ is transformed where $M(k,B)$ is the first order PT for the spheroid before it is being rotated and $M(k,B')$ is the first order PT for the spheroid after it is being rotated.

4. RESULTS

In this section, we will present some results regarding the transformation of the first order PT for spheroid. First of all, by using (1) and (3), we compute the first order PT for prolate spheroid B where a, b and k are fixed to $a = 2 > b = 1$ and $k = 1.5$. By assuming the prolate spheroid is rotated θ° counterclockwise a few times about *x*-axis, *y*-axis and *z*-axis, the first order PT for the spheroid after all rotations are computed according to Proposition 2. In this case, the following matrix (5), (6) and (7) are used for *R* in Proposition 2 where they are the usual rotation matrix for θ° around *x*-axis, *y*-axis and *z*-axis, denoted by $R_x(\theta^{\circ})$, $R_y(\theta^{\circ})$ and $R_z(\theta^{\circ})$.

$$R_x(\theta^\circ) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \tag{5}$$

$$R_y(\theta^\circ) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \tag{6}$$

$$R_z(\theta^\circ) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

The first order PT given in Table 4.1 are the transformation of the first order PT for prolate spheroid where $a = 2 > b = 1$ and $k = 1.5$ after the spheroid is rotated a few times about *x*-axis, *y*-axis and *z*-axis. The first order PT for the spheroid before it is being rotated is actually equal to the first order PT for the spheroid after it is rotated 360° about any axis. All the resulting transformation of the first order PT given in Table 4.1 also have only positive eigenvalues namely 3.8543 and 3.4715 to suggest that they are all positive definite as suggested by Proposition 1.

On the other hand, the first order PT given in Table 4.2 are the transformation of the first order PT for oblate spheroid where $a = 1 < b = 2$ and $k = 0.5$ after the spheroid is rotated a few times about *x*-axis, *y*-axis and *z*-axis. Similarly, the first order PT for the spheroid before it is being rotated is actually equal to the first order PT for the spheroid after it is rotated 360° about any axis. However, in contrast to the first order PT in Table 4.1, the resulting transformation of the first order PT given in Table 4.2 have only negative eigenvalues which are -11.3764 and – 9.5005 to suggest that they are all negative definite as suggested by Proposition 1.

Table 4.1 The first order PT for prolate spheroid $(a = 2 > b = 1$ and $k = 1.5)$ when the spheroid is rotated $\theta^\circ = 120$, 240 and 360 around (a) *x*-axis, (b) *y*-axis and (c) *z*-axis where, each $M(1.5, B')$ has two eigenvalues namely 3.8543 and 3.5715.

(a)

θ°	$M(1.5, B') = R_x(\theta^\circ)^T M(1.5, B) R_x(\theta^\circ)$
120	$\begin{bmatrix} 3.8543 & 0 & 0 \\ 0 & 3.5715 & 0 \\ 0 & 0 & 3.5715 \end{bmatrix}$
240	$\begin{bmatrix} 3.8543 & 0 & 0 \\ 0 & 3.5715 & 0 \\ 0 & 0 & 3.5715 \end{bmatrix}$
360	$\begin{bmatrix} 3.8543 & 0 & 0 \\ 0 & 3.5715 & 0 \\ 0 & 0 & 3.5715 \end{bmatrix}$

(b)

θ°	$M(1.5,B') = R_y(\theta^{\circ})^T M(1.5,B) R_y(\theta^{\circ})$
120	$\begin{bmatrix} 3.5672 & 0 & -0.1657 \\ 0 & 3.4715 & 0 \\ -0.1657 & 0 & 3.7586 \end{bmatrix}$
240	$\begin{bmatrix} 3.5672 & 0 & 0.1657 \\ 0 & 3.4715 & 0 \\ 0.1657 & 0 & 3.7586 \end{bmatrix}$
360	$\begin{bmatrix} 3.8543 & 0 & 0 \\ 0 & 3.5715 & 0 \\ 0 & 0 & 3.5715 \end{bmatrix}$

(c)

θ°	$M(1.5,B') = R_z(\theta^{\circ})^T M(1.5,B) R_z(\theta^{\circ})$
120	$\begin{bmatrix} 3.5672 & 0 & -0.1657 \\ 0 & 3.4715 & 0 \\ -0.1657 & 0 & 3.7586 \end{bmatrix}$
240	$\begin{bmatrix} 3.5672 & 0 & 0.1657 \\ 0 & 3.4715 & 0 \\ 0.1657 & 0 & 3.7586 \end{bmatrix}$
360	$\begin{bmatrix} 3.8543 & 0 & 0 \\ 0 & 3.5715 & 0 \\ 0 & 0 & 3.5715 \end{bmatrix}$

Table 4.2 The first order PT for prolate spheroid $(a = 1 < b = 2$ and $k = 0.5)$ when the spheroid is rotated $\theta^{\circ} = 120$, 240 and 360 around (a) *x*-axis, (b) *y*-axis and (c) *z*-axis where, each $M(0.5, B')$ has two eigenvalues namely –11.3764 and –9.5005.

(a)

θ°	$M(0.5,B') = R_x(\theta^{\circ})^T M(0.5,B) R_x(\theta^{\circ})$
120	$\begin{bmatrix} -11.3764 & 0 & 0 \\ 0 & -9.5005 & 0 \\ 0 & 0 & -9.5005 \end{bmatrix}$
240	$\begin{bmatrix} -11.3764 & 0 & 0 \\ 0 & -9.5005 & 0 \\ 0 & 0 & -9.5005 \end{bmatrix}$
360	$\begin{bmatrix} -11.3764 & 0 & 0 \\ 0 & -9.5005 & 0 \\ 0 & 0 & -9.5005 \end{bmatrix}$

(b)

θ°	$M(0.5,B')=R_y(\theta^{\circ})^T M(0.5,B)R_y(\theta^{\circ})$
120	$\begin{bmatrix} -9.9695 & 0 & 0.8123 \\ 0 & -9.5005 & 0 \\ 0.8123 & 0 & -10.9074 \end{bmatrix}$
240	$\begin{bmatrix} -9.9695 & 0 & -0.8123 \\ 0 & -9.5005 & 0 \\ -0.8123 & 0 & -10.9074 \end{bmatrix}$
360	$\begin{bmatrix} -11.3764 & 0 & 0 \\ 0 & -9.5005 & 0 \\ 0 & 0 & -9.5005 \end{bmatrix}$

(c)

θ°	$M(0.5,B')=R_z(\theta^{\circ})^T M(0.5,B)R_z(\theta^{\circ})$
120	$\begin{bmatrix} -9.9695 & -0.8123 & 0 \\ -0.8123 & -10.9074 & 0 \\ 0 & 0 & -9.5005 \end{bmatrix}$
240	$\begin{bmatrix} -9.9695 & 0.8123 & 0 \\ 0.8123 & -10.9074 & 0 \\ 0 & 0 & -9.5005 \end{bmatrix}$
360	$\begin{bmatrix} -11.3764 & 0 & 0 \\ 0 & -9.5005 & 0 \\ 0 & 0 & -9.5005 \end{bmatrix}$

Sometimes, we might want to identify an object independent of its orientation, for example, in the real applications such as metal detection for security screening and landmine clearance. Similarly, in this study, we might want to say that a spheroid before it is rotated and after it is rotated to be the same spheroid. In this case, we can make the identification based on the first order PT. Generally, the determinant of the first order PT of an object is the same with determinant of the first order PT of an object after it is being rotated, where, this is further explained in the next theorem.

Theorem 1 If M is the first order PT for a spheroid and M_R is the first order PT for the spheroid after it is rotated, then $\det(M)=\det(M_R)$.

Proof :

According to Proposition 2, $M_R = R^T M R$ where R is the appropriate rotation matrix. Since R is a unitary matrix transformation and by using the property of determinant, we have $\det(R)=1=\det(R^T)$.
Finally, by using the property of determinant again, it can be shown that $\det(M_R)=\det(R^T)\det(M)\det(R)$ which gives $\det(M)=\det(M_R)$.

In the results given in Table 4.1, all first order PT for prolate spheroid have determinant equal to 46.4505 whereas all first order PT for oblate spheroid given in Table 4.2 have determinant equal to -1026.8. Thus, the determinant of the first order PT for the spheroids can be used to show that they are all identical. This suggests a method to identify an object independent of its orientation based on the first order PT of an object.

5. CONCLUSION

In this paper, by evaluating the first order PT for two spheroids based on the explicit formula, we use matrix transformation to obtain the first order PT for the spheroids after they are being rotated. The resulting first order PT after transformation also satisfies the general characteristic of the first order PT which depends on the conductivity that is the material of the spheroids. We also discuss how to identify the spheroids before and after they are rotated based on their first order PT.

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