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Polynomial approximation of the numerical solutions of second order linear differential equations

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ABSTRACT

In this work the method of finding the solution of second order linear differential equations in the Chebyshev basis polynomial representation is studied and presented. In order to determine the coefficients of the solution of the differential equation which is assumed to have an orthogonal polynomial representation, the derivatives up to second order of each of the basis polynomials in its orthogonal representation has to be computed. The method reduces the problem into solving algebraic equations that approximate the coefficients of the particular integral. Comparison between the numerical solutions and the exact solutions of certain equations shows that in the case when the source function is not a polynomial equation, the results do not converge to the expected exact solution. Better approximation of the source function in terms of Chebyshev basis is required for the exponential or other trigonometric and transcendental functions.

Keywords: Chebyshev polynomial; Ordinary Differential Equation; Chebyshev Polynomial Approximation

1. INTRODUCTION

The study of differential equations originated in the investigation of laws that govern the physical world and were first solved by Sir Isaac Newton (1642-1727), who referred to them as 'fluxional' equations. The term 'differential' equation was introduced by Gottfried Leibniz (1646-1716) who, along with Newton, is credited with inventing the calculus. Many of the techniques for solving differential equations were known to mathematicians of the seventeenth century, but it was not until the nineteenth century that Augustin-Louis Cauchy (1789-1857) developed a general theory for differential equations that was independent of physical phenomena. We present here a method that can be applied to approximate the solution of second order linear differential equations of the form

$$a(x)\frac{d^2y}{dx^2}+b(x)\frac{dy}{dx}+c(x)y=g(x) \tag{1}$$

such that the integral is in the form of the Chebyshev polynomial basis. The source function $g(x)$ is therefore approximated using the Chebyshev polynomials.

2. APPROXIMATING TO SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS.

In 1992, J.C Butcher investigated the role of orthogonal polynomials in numerical ordinary differential equations [1]. Moreover, the study of finite difference approximation to ordinary differential equation has been done by M.R. Osborne in 1964. According to him, the method is given for the construction of finite difference approximations to ordinary linear differential equations, based on the assumption that the desired solution can be adequately represented by a certain interpolation polynomial [2]. Later in 2008, Lara et.al had investigated on the approximation to solutions of linear ordinary differential equations by cubic interpolation. They presented a method of integration for non-autonomous non-homogeneous systems of linear ordinary differential equations (ODES), which is based in both, the cubic polynomial segmentary interpolation and the minimal square method [3]. Recently, in 2013, Amber Sumner Robertson has investigated on the Chebyshev polynomial approximation to solution of ODES. He developed a method for finding approximate particular solutions for second order ordinary differential equation [4].

3. PRELIMINARY CONCEPTS

In order to approximate the solution of second order linear differential equations using the Chebyshev polynomials, the Chebyshev polynomials of the first kind is first presented. Refering to [5], The Chebyshev polynomials $T_n(x)$ of the first kind is a polynomial in x of degree n, defined by the relation

$$T_n(x)=\cos n\theta \text{ when } x=\cos\theta.$$

3.1 Generating Chebyshev Polynomials

According to De Moivre's Theorem, $\cos n\theta$ is a polynomial of degree n in $\cos\theta$.

$$\begin{aligned}
&\cos 0\theta = 1\\
&\cos 1\theta = \cos\theta\\
&\cos 2\theta = 2\cos^2\theta - 1\\
&\cos 3\theta = 4\cos^3\theta - 3\cos\theta\\
&\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1\\
&\dots
\end{aligned}$$

Hence, the first few Chebyshev polynomial basis are given by

$$\begin{aligned}
&T_0(x) = 1\\
&T_1(x) = \cos\theta = x\\
&T_2(x) = \cos 2\theta = 2\cos^2\theta - 1 = 2x^2 - 1\\
&T_3(x) = \cos 3\theta = 4\cos^3\theta - 3\cos\theta = 4x^3 - 3x\\
&\dots
\end{aligned}$$

By combining the trigonometric identity

$$\cos n\theta + \cos(n-2)\theta = 2\cos\theta\cos(n-1)\theta.$$

Hence,the fundamental recurrence relation for Chebyshev polynomials

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \;\; n = 2,3,\dots,$$

with the initial condition $T_0(x) = 1$, $T_1(x) = -1$. The plot of Chebyshev polynomials for $n = 0,1,2,3,4,5$ can be seen in Figure 1 below.

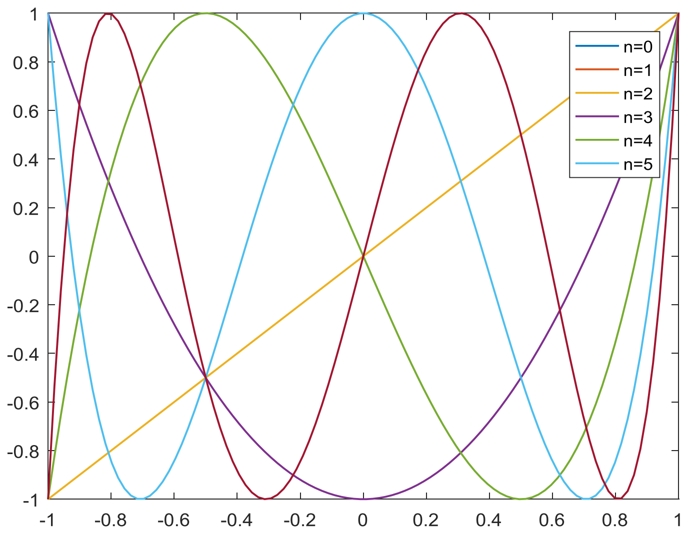

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Figure 1 Plot of Chebyshev Polynomials

3.2 APPROXIMATION OF FUNCTIONS USING CHEBYSHEV POLYNOMIALS

There are many related works that requires computations in the orthogonal polynomial basis since the method of transformation from the orthogonal basis polynomials is sometimes ill-conditioned. In this work, we investigate on the application of orthogonal polynomials to approximate the source function and solution of second order ordinary differential equations (ODES). Firstly, the source function $g(x)$ of equation (1) needs to be approximated using

Chebyshev polynomials. As we know, $T_n(x)$ denotes the Chebyshev polynomial of degree n where it has n roots which also known as Chebyshev nodes [6]. Then, these nodes can be calculated by the formula

$$x_i = \cos\left(\frac{i-\frac{1}{2}}{n}\right)\pi \text{ for } i = 1,2,\ldots,n+1.$$

A function $g(x)$ can be approximated by an n-th degree polynomial $P_n(x)$ expressed in terms of $T_{0,\ldots,}T_n$,

$$f(x) \approx P_n(x) = C_0T_0(x) + C_1T_1(x) + \ldots + C_nT_n(x) - \frac{1}{2}C_0,$$

where

$$C_j = \frac{2}{n}\sum_{k=1}^{n+1} f(x_k)T_j(x_k) \ , \quad j = 0,1,\ldots,n.$$

x_k , $k = 1,\ldots,n+1$ are zeros of T_{n+1}.
Since

$$T_j(x) = \cos(j\cos^{-1}x),$$

we have

$$T_j(x_k) = \cos(j\cos^{-1}x_k)$$
$$= \cos\left(\frac{j(k-\frac{1}{2})}{n+1}\pi\right)$$

Figure 2 give the functions used in the approximation of the Chebyshev basis polynomials using MATLAB.

```
%function approximate
n=3;
for i=1:n+1
x(i)=cos(((i-(1/2))/n)*pi);
end
for k=1:n+1
f(k)=%any function;
end
for j=1:n+1
for k=1:n+1
        T(j,k)=cos(((j-1)*(k-(1/2))/(n+1))*pi);
end
end
for j=1:n+1
sumA=0;
for k=1:n+1
sumA=sumA+(f(k)*T(j,k));
    C(j)=sumA*(2/n);

end
End
```

Figure 2: Function Approximation using Matlab Software

4. MATHEMATICAL FORMULATION

We apply the method for solving nonhomogeneous linear equations. As in the work of [4], we assume the particular integral to be in the form

$$y_p = \sum_{j=0}^{m} q_j T_j(x). \tag{2}$$

The coefficients q_j in equation (2) need to be determined. Since we have approximate the source function $g(x) \approx P_n(x)$. Let $m=n$ if $c \neq 0$ in equation (2) ; $m = n+1$ if $c = 0, b \neq 0$; $m = n+2$ if $c = 0, b = 0$. By using method of reduction order, the derivatives of particular integral is determined:

$$y'_p(x) = \sum_{j=0}^{m} q_j T'(x), \tag{3}$$

$$y''_p(x) = \sum_{j=0}^{m} q_j T''_j(x) \tag{4}$$

Substituting equation (2), (3) and (4) into equation (1) gives the equation

$$a\sum_{j=0}^{m} q_j T''_j(x) + b\sum_{j=0}^{m} q_j T'_j(x) + c\sum_{j=0}^{m} q_j T_j(x) = P_n(x) \tag{5}$$

To form a linear combination of equations of the form (5), $T''(x)$ and $T'(x)$ need to be determined in terms of Chebyshev polynomials basis. From the definition of Chebyshev polynomials, the monomial x^n can also be represented in the Chebyshev basis [5] as follows

$$\begin{aligned}
x^0 &= T_0 \\
x^1 &= T_1 \\
x^2 &= \frac{1}{2}(T_0 + T_2) \\
x^3 &= \frac{1}{4}(3T_1 + T_3) \\
x^4 &= \frac{1}{8}(3T_0 + 4T_2 + T_4) \\
x^5 &= \frac{1}{16}(10T_1 + 5T_3 + T_5) \\
&\vdots
\end{aligned}$$

where $x^n = 2^{1-n} \sum_{k=0}^{\lfloor n/2 \rfloor}{}' \binom{n}{k} T_{n-2k}(x).$ Unlike the approach in [4], the power of x relative to the Chebyshev basis is computed and substituted into the Chebyshev derivative series. Thus, the first derivatives of Chebyshev polynomials in terms of $T_n(x)$ are given as:

$$\begin{aligned}
T'_0(x) &= 0 \\
T'_1(x) &= 1 \\
T'_2(x) &= 4T_1(x) \\
T'_3(x) &= 6T_2(x) + 3T_0(x) \\
&\cdots
\end{aligned}$$

The list of second derivatives in terms of $T_n(x)$

$$T_0^{''}(x) = 0$$
$$T_1^{''}(x) = 0$$
$$T_2^{''}(x) = 4T_0$$
$$T_3^{''}(x) = 24T_1(x)$$
$$\dots$$

The coefficients of q_j in equation (5) can then be easily determined by comparing the equations and solve by using backward substitution. This leads to the approximation of the integral and the complimentary function y_c. The examples given in Section 5 below demonstrate the method employed. Comparison between the numerical solutions and exact solution are made when the initial value problem is applied on each problem.

5. EXAMPLES OF APPROXIMATION

Example 1 Consider the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2x^2 - 1.$$

with initial condition $y(0) = 1$ and $y^{'}(0) = 2$. Firstly, the right-hand side function of the differential equation needs to be approximated. The Chebyshev polynomial representation for $g(x)$ is $g(x) = \sum_{j=0}^{2} p_j T_j(x)$. By expanding the equation,

$$g(x) = p_0 T_0 + p_1 T_1 + p_2 T_2$$

Substitute the first three Chebyshev polynomials into this equation gives

$$\begin{aligned} g(x) &= p_0(1) + p_1(x) + p_2(2x^2 - 1) \\ &= (p_0 - p_2)x^0 + p_1 x^1 + 2p_2 x^2, \end{aligned} \tag{6}$$

by comparing the source function with equation (6) gives the coefficients p_j. Hence the coefficients of p_j for $j = 0,1,2$ are $p_0 = 0$, $p_1 = 0$, $p_2 = 1$. Therefore, the Chebyshev approximation of function $g(x) = 2x^2 - 1$.

$$g(x) \approx P_2(x) = T_2. \tag{7}$$

Then, we look for particular solution

$$\begin{aligned} y_p &= \sum_{j=0}^{3} q_j T_j(x) \\ &= q_0 T_0(x) + q_1 T_1(x) + q_2 T_2(x) + q_3 T_3(x) \end{aligned}$$

By reduction of the particular solution, we obtain y'_p and $y''p$. To get the coefficients q_j , substitute $T_j^{'}(x)$ and $T^{''}(x)$ in terms of $T(x)$ into y'_p and y''_p.

$$y_p^{'} = \sum_{j=0}^{3} q_j T_j^{'}(x) = (q_1 + 3q_3)T_0(x) + 4q_2 T_1(x) + 6q_3 T_2(x)$$
$$y_p^{''} = \sum_{j=0}^{3} q_j T_j^{''}(x) = 4q_2 T_0(x) + 24q_3 T_1(x).$$

Substitute the above equation into the original equation gives,

$$y_p'' - 2y_p' + y_p = (4q_0 - 2q_1 - 6q_3 + q_0)T_0(x) + (24q_3 - 8q_2 + q_1)T_1(x) + (-6q_3 + q_2)T_2(x) + q_3. \quad (8)$$

By comparing equation (7) with equation (8), we obtain

$$\begin{aligned} 4q_2 - 2q_1 - 6q_3 + q_0 &= 0 \\ 24q_3 - 8q_2 + q_1 &= 0 \\ -6q_3 + q_2 &= 1 \\ q_3 &= 0 \end{aligned}$$

This yields, $q_0 = 12$, $q_1 = 8$, $q_2 = 1$, $q_3 = 0$. Therefore, the Chebyshev approximation particular solution is $y_p = 12T_0(x) + 8T_1(x) + T_2(x)$. Next, the complimentary function is solved and the initial value problem is inserted into this problem to obtain the constants in the function. The numerical solution is approximated as

$$\tilde{y}(x) = (4x - 10)e^x + 12T_0(x) + 8T_1(x) + T_2(x).$$

The exact solution of the problem is

$$y(x) = (4x - 10)e^x + 2x^2 + 8x + 11.$$

After substituting the initial values, the constant values in the complimentary function of the corresponding homogeneous part tend to be close to the exact solution.

Example 2 Consider the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 6e^x,$$

with initial condition $y(0) = 1$ and $y'(0) = 2$. The source function is approximated using the Maclaurin series as

$$6e^x = 6 + 6x + 3x^2 + x^3 + \frac{x^5}{20} + \frac{x^6}{120} + \ldots$$

Solving the equation using the method described in Example 1 gives the approximation of the particular integral

$$y_p = \frac{93}{2}T_0(x) + \frac{99}{4}T_1(x) + 3T_2(x) + \frac{1}{4}T_3(x),$$

and the complimentary function is computed as $y_c = (-\frac{87}{2} + \frac{43}{2}x)e^x$. The numerical solution is therefore approximated as $\tilde{y} = (-\frac{87}{2} + \frac{43}{2}x)e^x + y_p$. On the other hand the exact solution is given by $y = (3x^2 + 2x)e^x$.

6. CONCLUSION

The source function in the first example is a polynomial function while the second example is the exponential function. In the second example it appears that the numerical solution of this problem physically does not equal to the exact solution. Two stages of approximation are involved in this case. Firstly, to approximate the exponential function of power series polynomials and then this followed by finding its approximation in terms of the Chebyshev basis polynomials. For this case the method applied in this work seems to be far from accurate, as compared to the case when the source function is a power series polynomial. Better approximation of the source function in terms of Chebyshev basis is required for the exponential or other trigonometric and transcendental functions.

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